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On the Variety Generated by Bounded Pseudo-BCK-Algebras

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Abstract In the paper we prove that the equational class $\mathcal{V}(bp\mathbb{BCK})$ generated by the class $bp\mathbb{BCK}$ of all bounded pseudo-BCK-algebras is generated by its simple members. As a matter of fact, we prove that simple members of $\mathcal{V}(bp\mathbb{BCK})$ just coincide with relative simple bounded pBCK-algebras. Moreover, as a byproduct we show that every simple bounded pBCK-algebra can be embedded into a simple integral residuated lattice.

Keywords Pseudo-BCK-algebra · Variety · Simple algebra · Relative simple algebra

1 Introduction

Pseudo-BCK-algebras were introduced by G. Georgescu and A. Iorgulescu as a non-commutative generalization of BCK-algebras [6]. Recall that BCK-algebras were introduced by K. Iséki [8] as an algebraic semantics of Meredith's implicational calculus, see also [10]. Bounded BCK-algebras were also treated by Iséki [9] as BCK-algebras with an additional constant $\perp$ interpreted as the lower bound. In fact they are the algebraic counterpart of BCK-calculus plus a negation satisfying Duns Scoto law. For more details we also refer the reader to the comprehensive monograph [12] or the papers [1, 2] and [19].

Pseudo-BCK-algebras are known to be one of the key structures with respect to many-valued reasoning. Among others, this class contains the implication reduct of MV-algebras, being equivalent algebraic semantics of many-valued Łukasiewicz logic. As shown by Z. Riečanová [16], lattice effect algebras are pastings of MV-effect algebras.

Analogously, pseudo-MV-algebras, a non-commutative version of MV-algebras, introduced by G. Georgescu and A. Iorgulescu [5] and independently (under the name GMV-algebras) by J. Rachůnek [15], play an important role in studying structure of pseudo-effect

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algebras. These were introduced by A. Dvurečenskij and T. Vetterlein [3, 4] as a basic structure for mathematical foundations of quantum mechanics.

In the same way, pseudo-BCK-algebras can be considered as an algebraic semantics of so-called pseudo-BCK-logic, introduced and studied by J. Kühr. Bounded pseudo-BCK-algebras form a quasivariety, which is not a variety, and hence are not definable by means of equations. Thus bounded pseudo-BCK-logic is algebraizable (in the sense of Block and Pigozzi) having the quasivariety of bounded pseudo-BCK-algebras as its algebraic semantics.

Based on the results of H. Ono and T. Kowalski [11], Gispert and Torrens proved that the variety $\mathcal{V}(b\mathbb{BCK})$ generated by the class $b\mathbb{BCK}$ of bounded BCK-algebras is generated by its finite simple members. The aim of this paper is to prove a similar statement for the class of bounded pseudo-BCK-algebras. Although the argumentation is in both cases very similar, one has to stress that the description of relative congruences is in the non-commutative case much more complicated. Moreover, as shown below, the class of bounded pseudo-BCK-algebras is not hereditary simple. The paper uses the fact that bounded pseudo-BCK-algebras are the $\{\to, \rightsquigarrow, \bot, \top\}$-subreducts of bounded integral residuated lattices.

In what follows we donote by $p\mathbb{BCK}$ the class of pseudo-BCK-algebras, and by $bp\mathbb{BCK}$ the class of all bounded pseudo-BCK-algebras. After repeating some well-known facts on pseudo-BCK-algebras, we show that simple algebras in $\mathcal{V}(bp\mathbb{BCK})$ are just relative simple bpBCK-algebras (see the definition below). Using the relationship between pseudo-BCK-algebras and bounded residuated lattices and the result of Takamura [17] on free bounded integral residuated lattices we show that the free members of $\mathcal{V}(bp\mathbb{BCK})$ are generated by its simple algebras. On the other hand, we present a quasi-identity which is valid for all simple bpBCK-algebras but which is not valid in $bp\mathbb{BCK}$. In other words, $bp\mathbb{BCK}$ is not as a quasivariety generated by its simple members.

For basic properties of pBCK-algebras we refer to [6]. Recall that a quasivariety (variety) is a class of algebras of the same type (or language), axiomatized by a set of quasi-identities (identities). A subquasivariety $\mathcal{R}$ of a quasivariety $\mathcal{Q}$ is a *relative subquasivariety* of $\mathcal{Q}$ if it can be defined by adding identities to an axiomatization of $\mathcal{Q}$. As usual, by $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_U$ we denote the operators of taking isomorphic images, subalgebras, direct products and ultraproducts, respectively. As it is well known $\mathbb{ISPP}_U(\mathcal{K})$ ($\mathbb{HSP}(\mathcal{K})$) is the smallest quasivariety (variety) containing the class $\mathcal{K}$.

Let $\mathcal{K}$ be a class of algebras and $\mathbf{A}$ an algebra, all of the same type. A congruence Θ on $\mathbf{A}$ is said to be a *congruence relative to* $\mathcal{K}$ (or a $\mathcal{K}$-*congruence*) if the quotient $\mathbf{A}/\Theta \in \mathcal{K}$. It is well known that if a class $\mathcal{K}$ is a quasivariety, then the set $Con_{\mathcal{K}}\mathbf{A}$ of all $\mathcal{K}$-congruences on $\mathbf{A}$ forms an algebraic lattice with respect to set inclusion. In this case we always have the identity $\Delta_{\mathbf{A}}$ and $\nabla_{\mathbf{A}} = A^2$ in $Con_{\mathcal{K}}\mathbf{A}$. We say that $\mathbf{A}$ is $\mathcal{K}$-*simple* (or, equivalently, *relative simple* with respect to $\mathcal{K}$) provided $Con_{\mathcal{K}}\mathbf{A} = \{\Delta_{\mathbf{A}}, \nabla_{\mathbf{A}}\}$. Clearly, for a non-trivial algebra $\mathbf{A}$ and $\Theta \in Con_{\mathcal{K}}\mathbf{A}$, $\mathbf{A}/\Theta$ is $\mathcal{K}$-simple if and only if Θ is $\mathcal{K}$-*maximal*, i.e. maximal in $(Con_{\mathcal{K}}\mathbf{A} \setminus \{\nabla_{\mathbf{A}}\}, \subseteq)$.

As usual by $\Theta(a, b)$ we denote the least congruence relation containg the pair $(a, b) \in A^2$.

2 Bounded Pseudo-BCK-Algebras

An algebra $\mathbf{A} = (A, \to, \rightsquigarrow, \top)$ of type $(2, 2, 0)$ is called a *pseudo*-BCK-*algebra* (pBCK-algebra for short) [13] provided the following identities and quasi-identities are satisfied in $\mathbf{A}$:

(1) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = \top$
(2) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = \top$
(3) $x = \top \to x = \top \rightsquigarrow x,\ x \to \top = \top$
(4) $(x \to y = y \to x = \top) \Rightarrow x = y$.

If, moreover, there is $\bot \in A$ such that **A** satisfies also

(5) $\bot \to x = \top$,

then **A** is called *bounded* (bpBCK-*algebra* for short).

The class $bp\mathbb{BCK}$ of all bounded pBCK-algebras is considered to be of the type $(2, 2, 0, 0)$. Clearly, the $\{\to, \rightsquigarrow, \top\}$-reduct of any bpBCK-algebra is a pBCK-algebra. The following properties are known to hold for pBCK-algebras, see e.g. [6]:

Lemma 1 *Every* pBCK-*algebra* **A** *satisfies the following identities*:

(6) $(x \to y) \to [(z \to x) \to (z \to y)] = \top$
(7) $x \rightsquigarrow \top = \top$
(8) $x \to x = x \rightsquigarrow x = \top$
(9) $x \to (y \to x) = x \rightsquigarrow (y \to x) = x \rightsquigarrow (y \rightsquigarrow x) = \top$
(10) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$
(11) $x \to ((x \to y) \rightsquigarrow y) = x \rightsquigarrow ((x \to y) \rightsquigarrow y) = x \rightsquigarrow ((x \rightsquigarrow y) \to y) = x \to ((x \rightsquigarrow y) \to y) = \top$
(12) $(x \to y) \to [(x_1 \to \cdots \to (x_n \to x) \cdots) \to (x_1 \to \cdots \to (x_n \to y) \cdots)] = \top$.

Moreover, if **A** *is bounded, then*

(13) $\bot \rightsquigarrow x = \top$.

As a corollary we obtain

Corollary 1 *For any* $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$ *the identities* (1)–(4) *and* (6)–(13) *hold. Additionally,* **A** *satisfies the following quasi-identities*:

(14) $x \to y = \top$ *iff* $x \rightsquigarrow y = \top$
(15) *If* $x \to y = \top$ *then*
  (a) $(z \to x) \to (z \to y) = \top$
  (b) $(z \to x) \rightsquigarrow (z \to y) = \top$
  (c) $(z \rightsquigarrow x) \to (z \rightsquigarrow y) = \top$
  (d) $(z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y) = \top$
  (e) $(y \to z) \to (x \to z) = \top$
  (f) $(y \to z) \rightsquigarrow (x \to z) = \top$
  (g) $(y \rightsquigarrow z) \to (x \rightsquigarrow z) = \top$
  (h) $(y \rightsquigarrow z) \rightsquigarrow (x \rightsquigarrow z) = \top$.

Moreover, (a) $\Leftrightarrow$ (b), (c) $\Leftrightarrow$ (d), (e) $\Leftrightarrow$ (f), (g) $\Leftrightarrow$ (h).

Proof (14) follows from (3) and (11). Further, if $x \to y = \top$ then according to (3) and (6) we obtain $(z \to x) \to (z \to y) = \top$. The rest is a direct conclusion of (1)–(3) and (14). □

The relation $\le$ defined on a carrier A of a pBCK-algebra **A** by

$$x \le y \quad \text{iff} \quad x \to y = \top \quad (\text{iff } x \rightsquigarrow y = \top)$$

is a partial order on A, called the *natural order* of $\mathbf{A}$.

By definition, both classes $p\mathbb{BCK}$ and $bp\mathbb{BCK}$ are proper quasivarieties, i.e. they are not varieties. This simply follows by the fact that the class $\mathbb{BCK}$ of BCK-algebras can be considered as a subclass of $p\mathbb{BCK}$. Namely, BCK-algebras are just pBCK-algebras for which $\to$ and $\rightsquigarrow$ coincide. The fact that $\mathbb{BCK}$ is not a variety was proved by Wroński while the bounded case is shown in [18], where a bounded linearly ordered BCK-algebra having a homomorphic image that is not a bounded BCK-algebra is presented.

Clearly, for any $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$ we have $Con\mathbf{A} = Con(\mathbf{A} \restriction \{\to, \rightsquigarrow, \top\})$. Given $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$, we start with the description of $Con_{bp\mathbb{BCK}}\mathbf{A}$. For this, we need the following concepts:

A subset $F \subseteq A$ is called an *implication filter* (*i-filter* for short) on $\mathbf{A}$ if

(i) $\top \in F$
(ii) $x, x \to y \in F \Rightarrow y \in F$.

A filter F is called *compatible* whenever

(iii) $\forall x, y \in A : x \to y \in F$ iff $x \rightsquigarrow y \in F$.

Lemma 2 *For any* $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$ *there is a* 1-1 *correspondence between* $Con_{bp\mathbb{BCK}}\mathbf{A}$ *and the set of compatible* i*-filters of* $\mathbf{A}$ *given by*

$$\Theta \longmapsto \top/\Theta,$$
$$F \longmapsto \Theta_F = \{(x, y) \in A^2 : x \to y \in F \ \&\ y \to x \in F\}.$$

Proof Given $\Theta \in Con_{bp\mathbb{BCK}}\mathbf{A}$, then $\top/\Theta$ is a compatible i-filter of $\mathbf{A}$. Indeed, $x, x \to y \in \top/\Theta$ yields $(x, \top) \in \Theta$, $(x \to y, \top) \in \Theta$, thus $(y, \top) = (\top \to y, x \to y) \in \Theta$. Further, let $x \to y \in \top/\Theta$. Then by (11) and (15g) $x \to ((x \to y) \rightsquigarrow y) = \top$, $(((x \to y) \rightsquigarrow y) \rightsquigarrow y) \to (x \rightsquigarrow y) = \top$, thus $x \to y \in \top/\Theta$ gives due to (3) and (8) $x \rightsquigarrow y \in \top/\Theta$. Also conversely, $x \rightsquigarrow y \in \top/\Theta$ yields $x \to y \in \top/\Theta$ in a similar way. Altogether, $\top/\Theta$ is a compatible i-filter of $\mathbf{A}$.

Conversely, assume that F is a compatible i-filter of $\mathbf{A}$. Clearly, Θ_F is reflexive in view of (8) and symmetric by the definition. To prove its transitivity, let $(a, b), (b, c) \in \Theta_F$. Then $c \to b \in F$, hence $(c \to b) \to ((b \to a) \rightsquigarrow (c \to a)) = \top \in F$ due to (1) and (14), and thus (ii) gives $(b \to a) \rightsquigarrow (c \to a) \in F$. Since F is compatible, we have also $(b \to a) \to (c \to a) \in F$ and applying (ii) once more, we conclude $c \to a \in F$. Analogously, $a \to c \in F$, verifying transitivity of Θ_F.

Let us prove that Θ_F is compatible with respect to "$\to$": assume (a, b), $(c, d) \in \Theta_F$. Then $\top = (c \to d) \to ((b \to c) \to (b \to d)) \in F$ by (6), thus according to (ii) $(b \to c) \to (b \to d) \in F$. Interchanging d and c we obtain $(b \to d) \to (b \to c) \in F$, and hence $(b \to c, b \to d) \in \Theta_F$. Further, (1) and (14) give $(a \to b) \to ((b \to c) \rightsquigarrow (a \to c)) = \top \in F$, thus by (ii) $(b \to c) \rightsquigarrow (a \to c) \in F$. Again, by using compatibility of F, also $(b \to c) \to (a \to c) \in F$. Similarly $(a \to c) \to (b \to c) \in F$, i.e. $(a \to c, b \to c) \in \Theta_F$. Finally, transitivity of Θ_F and the above properties ensure the compatibility of Θ_F with respect to "$\to$". Analogously, Θ_F is compatible with respect to "$\rightsquigarrow$" and $\Theta_F \in Con\mathbf{A}$.

To see that $\Theta_F \in Con_{bp\mathbb{BCK}}\mathbf{A}$, let

$$x/\Theta_F \to y/\Theta_F = \top/\Theta_F = y/\Theta_F \to x/\Theta_F.$$

Then $(x \to y, \top), (y \to x, \top) \in \Theta_F$, thus

$$\top \to (x \to y) = x \to y \in F \quad \text{and} \quad \top \to (y \to x) = y \to x \in F,$$

which gives $x/\Theta_F = y/\Theta_F$.

Conversely, if $\Theta \in Con_{bp\mathbb{BCK}}\mathbf{A}$, then $(x, y) \in \Theta_{\top/\Theta}$ iff $x \to y, y \to x \in \top/\Theta$ iff $x/\Theta \to y/\Theta = \top/\Theta = y/\Theta \to x/\Theta$. Since $\mathbf{A}/\Theta \in bp\mathbb{B}CK$, we deduce $x/\Theta = y/\Theta$, i.e. $(x, y) \in \Theta$. Hence $\Theta_{\top/\Theta} = \Theta$. Analogously $F_{\Theta_F} = F$. □

Lemma 3 *Let* $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$ *be a non-trivial algebra* (*i.e.*, $|A| > 1$). *Then*

(i) $\{\top\}$ *is the least compatible* i*-filter of* $\mathbf{A}$,
(ii) $\mathbf{A} \in bp\mathbb{BCK}$ *iff* $\Theta_{\{\top\}} = \Delta_{\mathbf{A}}$,
(iii) $\Theta(\bot, \top) = \nabla_{\mathbf{A}}$.

Proof (i) Clearly, for $F = \{\top\}$ we have $\top \in F$ and $x, x \to y \in F$ yield $\top = x \to y = \top \to y = y$ by (3). That F is compatible is seen from (14).

(ii) Follows directly from the definition of Θ_F.

(iii) Given $a \in A$, then $(a, \top) = (\top \to a, \bot \to a) \in \Theta(\bot, \top)$ by (3) and (5). Hence $\Theta(\bot, \top) = \nabla_{\mathbf{A}}$. □

From Lemma 3(iii) it follows that $\Theta(\bot, \top) = \nabla_{\mathbf{A}}$ is a compact element of $Con\mathbf{A}$, thus any non-trivial algebra $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$ has maximal congruences (i.e., maximal elements of $(Con\mathbf{A} \setminus \{\nabla_{\mathbf{A}}\}, \subseteq)$). Evidently, since $\nabla_{\mathbf{A}} \in Con_{bp\mathbb{BCK}}\mathbf{A}$, the same holds for $bp\mathbb{BCK}$-congruences.

The following crucial lemma shows that in fact both maximal sets coincide:

Lemma 4 *Let* $\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK})$ *be a non-trivial algebra, and let* $\Theta \in Con\mathbf{A}$. *Then* Θ *is maximal iff it is* $bp\mathbb{BCK}$*-maximal.*

Proof First, $\mathbf{A}$ is non-trivial iff $\Theta_{\{\top\}} \neq \nabla_{\mathbf{A}}$. Indeed, $\Theta_{\{\top\}} = \nabla_{\mathbf{A}}$ iff $(a, \top) \in \Theta_{\{\top\}}$ for all $a \in A$ iff $a = \top \to a \in \{\top\}$, i.e. iff $A = \{\top\}$.

Let now Θ be a maximal congruence on $\mathbf{A}$. Then $\mathbf{A}/\Theta$ is simple and non-trivial algebra in $\mathcal{V}(bp\mathbb{BCK})$. Thus $Con(\mathbf{A}/\Theta) = \{\Delta_{\mathbf{A}/\Theta}, \nabla_{\mathbf{A}/\Theta}\}$, $\Delta_{\mathbf{A}/\Theta} \neq \nabla_{\mathbf{A}/\Theta}$. Since $\mathbf{A}/\Theta$ is non-trivial, by the above arguments we conclude $\Theta_{\{\top/\Theta\}} \neq \nabla_{\mathbf{A}/\Theta}$, hence $\Theta_{\{\top/\Theta\}} = \Delta_{\mathbf{A}/\Theta}$. But this yields $\mathbf{A}/\Theta \in bp\mathbb{BCK}$ in view of Lemma 4(ii). In other words, $\Theta \in Con_{bp\mathbb{BCK}}\mathbf{A}$.

Conversely, assume that Θ is a maximal $bp\mathbb{BCK}$-congruence of $\mathbf{A}$. Let $\phi \in Con\mathbf{A}$, $\Theta \subseteq \phi, \Theta \neq \phi$. If $\mathbf{A}/\phi$ is non-trivial, then $\mathbf{A}/\phi \notin bp\mathbb{BCK}$ due to the maximality of Θ. Applying Lemma 4(ii) we obtain $\Theta_{\{\top/\phi\}} \neq \Delta_{\mathbf{A}/\phi}$. Since $\{\top/\phi\}$ is a compatible i-filter of $\mathbf{A}/\phi$, we deduce $(\mathbf{A}/\phi)/\Theta_{\{\top/\phi\}} \in bp\mathbb{BCK}$ is a non-trivial algebra. But one can easily see that $(\mathbf{A}/\phi)/\Theta_{\{\top/\phi\}}$ is a homomorphic image of $\mathbf{A}/\Theta \in bp\mathbb{BCK}$, which contradicts the maximality of Θ. In conclusion $\phi = A^2$ and Θ is maximal in $Con\mathbf{A}$. □

For a class $\mathcal{K}$ of algebras of the same type denote by $\mathcal{K}_S$ its simple members and by $\mathcal{K}_{SS}$ its semisimple members, i.e. subdirect products of members of $\mathcal{K}_S$. As a corollary we obtain

Theorem 1

(i) $(bp\mathbb{BCK})_S = (\mathcal{V}(bp\mathbb{BCK}))_S = \{\mathbf{A} \in \mathcal{V}(bp\mathbb{BCK}) : \mathbf{A} \text{ is } bp\mathbb{BCK} - \text{simple}\}$
(ii) $(bp\mathbb{BCK})_{SS} = (\mathcal{V}(bp\mathbb{BCK}))_{SS} \subseteq \mathbb{ISP}((bp\mathbb{BCK})_S)$.

Remark In contrast to bounded BCK-algebras, pBCK-algebras *are not hereditary simple*. That means, subalgebras of simple pBCK-algebras need not be simple. This can be seen from the following example which due to C. van Alten:

Example 1 Consider the five-element residuated lattice (see the definition below) with linear order $1 > a > b > c > 0$ and monoid operation $\odot$ with the corresponding residua $\rightarrow$ and $\rightsquigarrow$ given by tables

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	b	c	a
b	0	c	0	0	b
c	0	c	0	0	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	b	1	1	b	1
c	b	1	1	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	b	1	1	b	1
c	b	1	1	1	1
1	0	a	b	c	1

One can easily verify that $(\{0, a, b, c, 1\}, \rightarrow, \rightsquigarrow)$ is a simple bpBCK-algebra having a non-simple subalgebra $\{0, a, 1\}$ as $\{1, a\}$ is its non-trivial compatible i-filter.

The above example also shows that the inclusion in Theorem 1(ii) can not be substituted by the equality.

3 $\mathcal{V}(bp\mathbb{BCK})$ is Generated by Simple bpBCK-Algebras

In the following we prove that the variety $\mathcal{V}(bp\mathbb{BCK})$ is generated by simple bpBCK-algebras. We use the results of J. Kühr [13, 14] and H. Takamura [17] on free residuated lattices. First, recall the relationship between these algebras and bpBCK-algebras.

A *bounded integral residuated lattice*, or *bounded residuated lattice* for short, is an algebra $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, \wedge, \vee, \perp, \top)$ of type $(2, 2, 2, 2, 2, 0, 0)$ such that $(A, \odot, \top)$ is a monoid (not necessarily commutative), $(A, \wedge, \vee, \perp, \top)$ is a bounded lattice, and the following residuation conditions hold:

$$x \odot y \le z \quad \Longleftrightarrow \quad x \le y \rightarrow z \quad \Longleftrightarrow \quad y \le x \rightsquigarrow z,$$

where $\le$ is the order induced by the lattice structure, called the *natural order* of $\mathbf{A}$, see [12].

The class of all bounded integral residuated lattices is a variety that we shall denote by $\mathbb{RL}$. It has been proved by J. Kühr [13] that $bp\mathbb{BCK}$ is just the class of all $\{\rightarrow, \rightsquigarrow, \perp, \top\}$-subreducts of $\mathbb{RL}$.

Moreover, for $\mathbf{R} \in \mathbb{RL}$ we have

$$Con\mathbf{R} = Con_{bp\mathbb{BCK}}(\mathbf{R} \restriction \{\rightarrow, \rightsquigarrow, \perp, \top\}). \qquad (*)$$

Indeed, if $\Theta \in Con\mathbf{R}$, then trivially $\Theta \in Con_{bp\mathbb{BCK}}(\mathbf{R} \restriction \{\rightarrow, \rightsquigarrow, \perp, \top\})$.

Conversely, let $\Theta \in Con_{bp\mathbb{BCK}}(\mathbf{R} \restriction \{\rightarrow, \rightsquigarrow, \perp, \top\})$. Since $\mathbf{R} \in \mathbb{RL}$, we have $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ for all $x, y, z \in R$.

Further, $(\mathbf{R} \restriction \{\rightarrow, \rightsquigarrow, \perp, \top\})/\Theta \in bp\mathbb{BCK}$, hence $a/\Theta \rightarrow b/\Theta = \top/\Theta = b/\Theta \rightarrow a/\Theta \Rightarrow a/\Theta = b/\Theta$ for all $a, b \in A$. In other words,

$$(a \rightarrow b, \top) \in \Theta \quad \text{and} \quad (b \rightarrow a, \top) \in \Theta \quad \Longrightarrow \quad (a, b) \in \Theta. \qquad (**)$$

We prove that Θ is compatible with respect to "$\odot$"; assume $(x, y), (u, v) \in \Theta$. Then $(x \odot u) \rightarrow (y \odot v) = x \rightarrow (u \rightarrow (y \odot v)) \equiv_\Theta y \rightarrow (v \rightarrow (y \odot v)) = (y \odot v) \rightarrow (y \odot v) = \top$. Then applying $(**)$ we obtain $(x \odot u, y \odot v) \in \Theta$, as desired.

In the following we investigate the connection between simple residuated lattices and simple bpBCK-algebras. Let us stress that due to Example 1, subreducts of simple residuated lattices need not be simple bpBCK-algebras. However, we are able to prove

Theorem 2 *The* $\{\to, \rightsquigarrow, \bot, \top\}$*-reducts of simple residuated lattices are simple* bpBCK-*algebras. Conversely, every simple* pbBCK-*algebra is a* $\{\to, \rightsquigarrow, \bot, \top\}$*-subreduct of a simple residuated lattice.*

Proof If $\mathbf{A} \in \mathbb{RL}$ is simple, then by the correspondence (∗), $\mathbf{A} \upharpoonright \{\to, \rightsquigarrow, \bot, \top\}$ is a $bp\mathbb{BCK}$-simple algebra; thus, by Theorem 1, $\mathbf{A} \upharpoonright \{\to, \rightsquigarrow, \bot, \top\}$ is simple in $bp\mathbb{BCK}$.

Conversely, let $\mathbf{B}$ be a simple algebra in $bp\mathbb{BCK}$ and let $\mathbf{R}$ be a residuated lattice for which $\mathbf{B}$ is its $\{\to, \rightsquigarrow, \bot, \top\}$-subreduct. Let M be a maximal compatible i-filter in $\mathbf{R}$; note that congruences of algebras in $\mathbb{RL}$ just correspond to their compatible i-filters. Clearly $M \cap B$ is a compatible i-filter of $\mathbf{B}$. Since M is proper, $\bot \notin M$. Again, by Theorem 1, $\mathbf{B}$ is $bp\mathbb{BCK}$-simple. But compatible i-filters correspond to $bp\mathbb{BCK}$-congruences, hence due to $\bot \notin M \cap B$ we deduce $M \cap B = \{\top\}$. Further, let $a, b \in B$, $a \neq b$. Then $a \to b \in B \setminus \{\top\}$ or $b \to a \in B \setminus \{\top\}$ (otherwise we would get $a = b$ as $\mathbf{B} \in bp\mathbb{BCK}$), thus $a/\Theta_M \neq b/\Theta_M$. Hence the mapping $a \longmapsto a/\Theta_M$ gives a bounded embedding of $\mathbf{B}$ into the $\{\to, \rightsquigarrow, \bot, \top\}$-subreduct of a simple bounded residuated lattice $\mathbf{R}/\Theta_M$. □

Given a class $\mathcal{K}$ of $\mathcal{L}$-algebras and X a set of variables, denote by $\mathcal{F}_{\mathcal{K}}(X)$ the $|X|$-free algebra in $\mathcal{K}$ (if it exists). It is well known that if $\mathcal{K}$ is a quasivariety, $\mathcal{F}_{\mathcal{K}}(X)$ exists for every $X \neq \emptyset$. Moreover, $\mathcal{F}_{\mathcal{K}}(X) = \mathcal{F}_{\mathcal{V}(\mathcal{K})}(X)$.

For two algebraic languages $\mathcal{L}$ and $\mathcal{L}'$ where $\mathcal{L}' \supseteq \mathcal{L}$ (i.e., $\mathcal{L}'$ is an expansion of $\mathcal{L}$) and for a class $\mathcal{K}$ of $\mathcal{L}'$-algebras, denote by $\mathcal{S}(\mathcal{K} \upharpoonright \mathcal{L})$ the class of $\mathcal{L}$-subreducts of elements of $\mathcal{K}$. The following holds [7]:

Proposition 1 *Let $\mathcal{K}$ has a free algebra $\mathcal{F}_{\mathcal{K}}(X)$ over X. Then $\mathcal{F}_{\mathcal{S}(\mathcal{K} \upharpoonright \mathcal{L})}(X)$ exists and it is the $\mathcal{L}$-subreduct of $\mathcal{F}_{\mathcal{K}}(X)$ generated by X.*

As an immediate corollary we obtain

Corollary 2 *For any set X, $\mathcal{F}_{bp\mathbb{BCK}}(X) = \mathcal{F}_{\mathcal{V}(bp\mathbb{BCK})}(X)$ is the $\{\to, \rightsquigarrow, \bot, \top\}$-subreduct of $\mathcal{F}_{\mathbb{RL}}(X)$.*

Proof Follows from Proposition 1 and from the fact $bp\mathbb{BCK} = \mathcal{S}(\mathbb{RL} \upharpoonright \{\to, \rightsquigarrow, \bot, \top\})$. □

H. Takamura [17] has shown that for any set X, $\mathcal{F}_{\mathbb{RL}}(X)$ is semisimple. Thus $\mathcal{F}_{\mathbb{RL}}(X)$ can be embedded into the product of simple residuated lattices $\{\mathbf{S}_i;\ i \in I\}$. According to Theorem 2, $\mathbf{S}_i \upharpoonright \{\to, \rightsquigarrow, \bot, \top\} \in (bp\mathbb{BCK})_S$ for every $i \in I$, thus $\mathcal{F}_{\mathbb{RL}}(X) \upharpoonright \{\to, \rightsquigarrow, \bot, \top\} \in \mathbb{ISP}((bp\mathbb{BCK})_S)$. Applying Corollary 2 we obtain

$$\mathcal{F}_{\mathcal{V}(bp\mathbb{BCK})}(X) \in \mathbb{S}(\mathcal{F}_{\mathbb{RL}}(X) \upharpoonright \{\to, \rightsquigarrow, \bot, \top\}) \subseteq \mathbb{ISP}((bp\mathbb{BCK})_S),$$

thus we have

Corollary 3 $\mathcal{V}(bp\mathbb{BCK}) = \mathcal{V}((bp\mathbb{BCK})_S)$.

The above corollary states that simple bpBCK-algebras generate the variety generated by the class $bp\mathbb{BCK}$. However, we can easily show that $bp\mathbb{BCK}$ is not as a quasivariety generated by the class $(bp\mathbb{BCK})_S$. In other words, we present the quasi-identity valid in $(bp\mathbb{BCK})_S$ which is not satisfied in $bp\mathbb{BCK}$. In fact, only a slight modification of a quasi-identity presented in [7] for the class $b\mathbb{BCK}$ is sufficient:

Lemma 5 *The quasi-identity*

$$(x \to y \approx x \rightsquigarrow y \text{ and } x \to \neg x \approx \neg x) \quad \Longrightarrow \quad \neg x \to x \approx x$$

holds in $(bp\mathbb{BCK})_S$ *but it does not hold in* $bp\mathbb{BCK}$.

Proof Let $\mathbf{A} \in bp\mathbb{BCK}$ satisfy the identities $x \to y \approx x \rightsquigarrow y$ and $x \to \neg x \approx \neg x$. Clearly, $x \to y \approx x \rightsquigarrow y$ means that $\mathbf{A} \in b\mathbb{BCK}$. According to [7], the quasi-identity $x \to \neg x \approx \neg x \Rightarrow \neg x \to x \approx x$ is valid in $(b\mathbb{BCK})_S$ but it is not valid in $b\mathbb{BCK}$. □

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